The Unified Model

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Abstract

A unified way to model derivative securities.

Every continuous time arbitrage-free model of instrument prices (X_t) with corresponding cash-flows (C_t) has the form

$$X_t D_t = M_t - \sum_{s \le t} C_s D_s$$

where (M_t) is a vector-valued martingale indexed by market instruments and (D_t) are positive, adapted functions. If the continuously compounded forward rate at t is f_t then $D_t = \exp(-\int_0^t f_s\,ds)$. If trading times are discrete, $T = \{t_j\}$, then $D_{t_j} = \exp(-\sum_{i<j} f_i\,\Delta t_i)$ where $\Delta t_i = t_{i+1} - t_i$ and f_i is the repurchase agreement rate over that interval.

For example, the Black-Scholes/Merton model for a bond and stock with no dividends is given by $M_t = (r, s\exp(\sigma B_t - \sigma^2 t/2))$ and $D_t = \exp(-\rho t)$ where (B_t) is standard Brownian motion. There is no need for self-financing portfolios, Ito's Lemma, much less partial differential equations when using the Unified Model.

The Unified Model provides a framework for a rigorous mathematical approach to understanding how to value, hedge, and manage risk using realistic trading conditions.

Introduction

The value of a barrier option in the Black-Scholes/Merton model that knocks in the *second* time the underlying hits the barrier is equal to the value of the option that knocks in the first time the underlying hits the barrier. In fact, the value is the same if it knocks in on the n-th time it hits the barrier for any $n > 0$! This is a mathematical artifact of Brownian motion having infinite total variation on any interval and the ridiculous notion that continuous time hedging is possible.

When a model in mathematical physics does not fit observations it indicates there is a flaw in the model. The Unified Model can be used to remedy the above flaw in the classical theory of mathematical finance. It places instrument prices and cash-flows on equal footing to clarify fundamental results like cost-of-carry, put-call parity, and the fact futures quotes are a martingale.

It also highlights the fundamental problem faced by all traders: when and how much to hedge.

Notation

If $\mathcal{A}$ is an algebra on the set Ω we write $X\colon \mathcal{A} \to \mathbf{R}$ to indicate $X\colon \Omega \to \mathbf{R}$ is $\mathcal{A}$-*measurable*. If $\mathcal{A}$ is finite then the atoms of $\mathcal{A}$ form a partition of Ω and being measurable is equivalent to being constant on atoms. In this case X is a function on the atoms of $\mathcal{A}$. The space of bounded $\mathcal{A}$-measurable functions is denoted $B(\mathcal{A})$.

A *filtration* on Ω indexed by $T \subseteq [0, \infty)$ is an increasing collection of algebras $(\mathcal{A})_{t\in T}$. The algebra $\mathcal{A}_t$ represents the information available at time t.

A process $M_t\colon \mathcal{A}_t \to \mathbf{R}$, $t \in T$, is a *martingale* if $M_t = E[M_u|\mathcal{A}_t] = E_t[M_u]$ for $t \le u$, where $E[X|\mathcal{A}]$ is the conditional expectation of the random variable X given the algebra $\mathcal{A}$.

A *stopping time* is a function $\tau\colon \Omega \to T$ satisfying $\{\tau \le t\} \in \mathcal{A}_t$ for all $t \in T$. The algebra $\mathcal{A}_\tau$ is the collection of subsets $E \subseteq \Omega$ with $E \cap \{\tau \le t\} \in \mathcal{A}_t$ for all $t \in T$. Stopping times can't peek into the future.

Unified Model

Let $T \subseteq [0, \infty)$ be the set of *trading times*. As is customary we assume a sample space, probability measure, and filtration are given, $\langle \Omega, P, (\mathcal{A}_t)_{t\in T}\rangle$.

Let I be the set of market *instruments* available for trading. Instrument *prices* are denoted by $X_t\colon \mathcal{A}_t \to \mathbf{R}^I$ and their corresponding *cash-flows* by $C_t\colon \mathcal{A}_t \to \mathbf{R}^I$, for $t \in T$. We assume, as is true in the real world, that prices and cash-flows are bounded.

Instrument trading is assumed to be perfectly liquid and divisible: every instrument can be bought or sold at the given price in any amount. Cash flows are associated with owning an instrument: stocks have dividends, bonds have coupons, futures have margin adjustments.

A *trading strategy* is a finite collection of strictly increasing stopping times (τ_j) and *trades* $\Gamma_j : \mathcal{A}_{\tau_j} \to \mathbf{R}^I$ indicating the number of shares to trade in each instrument. Trades accumulate to a *position* $\Delta_t = \sum_{\tau_j<t} \Gamma_j = \sum_{s<t} \Gamma_s$ where $\Gamma_s = \Gamma_j$ when $s = \tau_j$. Note that trades at time t are not included in the position at time t. It takes time for trades to settle before being included in the position.

The *value* (or *mark-to-market*) of a position at time t is $V_t = (\Delta_t + \Gamma_t)\cdot X_t$: how much you would get from liquidating the existing position and the trades just executed at price X_t, assuming that is possible. The *amount* generated by the trading strategy at time t is $A_t = \Delta_t \cdot C_t - \Gamma_t \cdot X_t$: you receive the cash flows associated with your existing position and pay for the trades just executed at the current market price.

A model is *arbitrage-free* if there is no trading strategy with $\sum_j \Gamma_j = 0$, $A_{\tau_0} > 0$, and $A_t \geq 0$ for $t > \tau_0$: it is impossible to make money on the first trade and never lose until the strategy is closed out.

Theorem. (Fundamental Theorem of Asset Pricing) *A model is arbitrage-free if and only if there exist deflators $D_t\colon \mathcal{A}_t \to (0,\infty)$, for $t \in T$, with*

$$X_t D_t = E[X_v D_v + \sum_{t<u\leq v} C_u D_u | \mathcal{A}_t].$$

If $C_t = 0$ for $t \in T$ then deflated prices are a martingale. If $E_t[X_v D_v] \to 0$ as $v \to \infty$ then deflated prices are the expected value of deflated future cash-flows, à la Dodd-Graham. We can assume $D_0 = 1$ by dividing all deflators by D_0.

One consequence of the displayed equation above and the definition of value and amount is

$$V_t D_t = E[V_v D_v + \sum_{t<u\leq v} A_u D_u | \mathcal{A}_t].$$

Note how value corresponds to prices and amount corresponds to cash-flows in the two formulas above. The second formula is the key to valuing derivatives. A derivative is a contract specifying payments at given times. If a trading strategy produces these payments as amounts then its value is given by this formula. Trading strategies create synthetic market instruments. Synthetic market instruments can become actual market instruments that are then included in I. The Unified Model can incorporate those without any changes.

Proof. If $u > t$ is sufficiently small then $\Delta_t + \Gamma_t = \Delta_u$ and $X_t D_t = E_t[(X_u + C_u)D_u]$. Since $V_t = (\Delta_t + \Gamma_t)\cdot X_t$

$$\begin{aligned}
V_t D_t &= (\Delta_t + \Gamma_t)\cdot X_t D_t \\
&= \Delta_u \cdot E_t[(X_u + C_u)D_u] \\
&= E_t[(\Delta_u \cdot X_u + \Delta_u \cdot C_u)D_u] \\
&= E_t[(\Delta_u \cdot X_u + \Gamma_u \cdot X_u + A_u)D_u] \\
&= E_t[(V_u + A_u)D_u]
\end{aligned}$$

where we use $\Delta_u \cdot C_u = \Gamma_u \cdot X_u + A_u$ and $(\Delta_u + \Gamma_u)\cdot X_u = V_u$ in the last two equalities respectively. The second displayed formula above follows by induction.

Assuming no arbitrage, $V_{\tau_0} D_{\tau_0} = E_{\tau_0}[\sum_{t>\tau_0} A_t D_t] \geq 0$. Since D_{τ_0} is positive and $V_{\tau_0} = \Gamma_{\tau_0} \cdot X_{\tau_0} = -A_{\tau_0}$ we have $A_{\tau_0} \leq 0$. This proves the “easy” direction of the theorem.

There is no need to prove the “hard” direction since we have a large supply of arbitrage free models. Every model of the form

$$X_t D_t = M_t - \sum_{s\le t} C_s D_s$$

where $M_t : \mathcal{A}_t \to \mathbf{R}^I$ is a martingale and $D_t : \mathcal{A}_t \to (0, \infty)$ is arbitrage-free.

$$\begin{aligned}
X_t D_t &= M_t - \sum_{s\le t} C_s D_s \\
&= E_t[M_v - \sum_{s\le t} C_s D_s] \\
&= E_t[M_v - \sum_{s\le v} C_s D_s + \sum_{t<u\le v} C_u D_u] \\
&= E_t[X_v D_v + \sum_{t<u\le v} C_u D_u].
\end{aligned}$$

Examples

We illustrate the Unified Model in particular cases.

Black-Scholes/Merton

The sample space is $\Omega = C[0, \infty)$, P is Wiener measure, and $\mathcal{A}_t$ is the smallest sigma-algebra for which $\{B_s : s \le t\}$ are measurable, where $B_t(\omega) = \omega(t)$ is standard Brownian motion.

The trading times are $T = [0, \infty)$ and the instruments are a bond with constant continuously compounded rate ρ and a stock paying no dividends with volatility σ. The Black-Scholes/Merton model is given by $M_t = (r, se^{\sigma B_t - \sigma^2 t/2})$ and $D_t = e^{-\rho t}$. It is trivial to extend this to the case when ρ is a function of time.

Note that the Unified Model does not require Ito’s Lemma or partial differential equations. Although some mathematicians who have invested their time learning these topics may be disappointed, their students will be glad these are no longer necessary.

Repurchase Agreement

In a discrete time model with $T = \{t_j\}$ a *repurchase agreement* at time t_j, has price $X_{t_j} = 1$ and cash flow $C_{t_{j+1}} = R_j$ where $R_j = \exp(f_j \Delta t_j)$ is the realized return for the *repo rate* f_j. The *canonical deflator* $D_{t_j} = 1/\Pi_{i<j} R_i = \exp(-\sum_{i<j} f_i\,\Delta_i)$ provides an arbitrage-free model for repos since $1\,D_{t_j} = E_{t_j}[R_j D_{t_{j+1}}]$.

The continuous time analog is $D_t = \exp(-\int_0^t f_s\,ds)$ where (f_t) is the *continuously compounded forward rate*.

Zero Coupon Bond

A *zero coupon bond* maturing at time u has a single cash flow $C_u = 1$ at time u. Its price at time t, $D_t(u)$, satisfies $D_t(u)D_t = E_t[1D_u]$ so $D_t(u) = E_t[D_u]/D_t$. The dynamics of all fixed income instruments are determined by the deflators: cash deposits, forward rate agreements, swaps, puts, floors, swaptions, etc.

Cost of Carry

A *forward* on an instrument S *expiring* at t with *strike* k has a single cash-flow $C_t = S_t - k$ at expiration. The *at-the-money forward* is the strike f that makes the forward price zero.

Consider any arbitrage-free model with $X_0 = (1, s, 0)$ and $X_t = (R, S_t, S_t - f)$. We have $(1, s, 0) = E[(R, S_t, S_t - f)D_t]$ assuming $D_0 = 1$. If R is constant, $1 = E[RD_t]$ so $E[D_t] = 1/R$. Since $s = E[S_tD_t]$ and $0 = E[(S_t - f)D_t]$ we have $0 = s - f/R$. The formula $Rs = f$ is the *cost of carry* and relates the *spot price* of S to its forward.

Put-Call Parity

A *put option* on an instrument S *expiring* at t with *strike* k has a single cash-flow $C_t = \max\{k - S_t, 0\}$ at expiration. A *call option* has a single cash-flow $C_t = \max\{S_t - k, 0\}$ at expiration. Note $\max\{S_t - k, 0\} - \max\{k - S_t, 0\} = S_t - k$.

Consider any one-period arbitrage-free model with $X_0 = (1, s, p, c)$ and $X_t + C_t = (R, S_t, \max\{k - S_t, 0\}, \max\{S_t - k, 0\})$. For any deflator with $D_0 = 1$ we have $(1, s, p, c) = E[(R, S_t, \max\{k - S_t, 0\}, \max\{S_t - k, 0\})D_t]$. Assuming R is constant, $1 = E[RD_t]$ so $E[D_t] = 1/R$. Since $p = E[\max\{k - S_t, 0\}D_t]$ and $c = E[\max\{S_t - k, 0\}D_t]$ we have $c - p = E[(S_t - k)D_t] = s - k/R$. This formula is called *put-call parity*. It holds for every arbitrage-free model and will be the first thing a trader tests when presented with a new model.

Futures

A *futures* on an instrument S *expiring* at t has a cash-flow at every *margin calculation date* $(t_j)_{j=0}^n$. The *futures quote* at expiration $t = t_n$ is $\Phi_n = S_t$. The cash-flow at time t_j is $C_{t_j} = \Phi_j - \Phi_{j-1}$, $1 \le j \le n$, where Φ_j is the futures quote at t_j.

The price of a futures is always zero. If the model is arbitrage-free then $0 = E_{t_{j-1}}[(\Phi_j - \Phi_{j-1})D_{t_j}]$. If D_{t_j} is $\mathcal{A}_{t_{j-1}}$ measurable then $\Phi_{j-1} = E_{t_{j-1}}[\Phi_j]$ so the futures quotes (Φ_j) are a martingale. This holds when $D_{t_j} = \exp(-\sum_{i<j} f_i\,\Delta_i)$.

American Option

An *American option* is an option that the holder can exercise at any time up to expiration. A call option on S expiring at t with strike k has a single cash-flow $C_\tau = \max\{S_\tau - k, 0\}$ at time τ where $\tau \leq t$ is chosen by the option holder.

The space of outcomes must include this possibility. Given a model for the underlying $\langle\Omega, P, (\mathcal{A}_t)\rangle$ let $\Omega' = \Omega \times [0, t]$ where (ω, τ) indicates the option is exercised at time τ given the underlying determined by ω.

The filtration must also be augmented. Let $\mathcal{T}_s$ be the smallest algebra on $[0, t]$ containing the singletons $\{u\}$ for $u \leq s$ and the set $(s, t]$. If $\tau \leq s$ then τ is known exactly, otherwise it is only known that $s < \tau \leq t$. The algebra $\mathcal{A}'_s = \mathcal{A}_s \times \mathcal{T}_s$ represents the information available at time s. Note that at time s it is known if $\tau = s$. The option holder decides when to exercise.

Extending the measure P on Ω to P' on Ω' while keeping the model arbitrage-free is not trivial. It would imply a solution to the American option pricing formula which (currently) does not have a closed form.

Remarks

Given a derivative paying $\bar{A}_j$ at times $\bar{t}_j$ how does one find a trading strategy (τ_j) and (Γ_j) with $A_t = \bar{A}_j$ at times $t = \bar{t}_j$ and zero otherwise?

The initial hedge is determined by $V_0 = E[\sum_{\bar{t}_j > 0} \bar{A}_j D_{\bar{t}_j}]$ which can be computed using the derivative payments specified in the contract and the deflators of the model. Since $V_0 = \Gamma_0 \cdot X_0$ we have $\Gamma_0 = dV_0/dX_0$ where the right-hand side is the Fréchet derivative of $V_0\colon \mathbf{R}^I \to \mathbf{R}$ with $X_0 \mapsto \Gamma_0 \cdot X_0$.

At any time t we have $V_t = E_t[\sum_{\bar{t}_j > t} \bar{A}_j D_{\bar{t}_j}]/D_t$ which can be computed using the specified derivative payments and the deflators. Since $V_t = (\Delta_t + \Gamma_t) \cdot X_t$ we have $\Delta_t + \Gamma_t = dV_t/dX_t$ where the right-hand side is the Fréchet derivative of $V_t\colon B(\mathcal{A}_t, \mathbf{R}^I) \to B(\mathcal{A}_t)$ with $X_t \mapsto (\Delta_t + \Gamma_t) \cdot X_t$.

This is classical Black-Scholes/Merton hedging with Δ being delta and Γ being gamma, however there is one major difference: there is no guarantee this hedge will replicate the option. As any trader knows after the second day on a trading floor, no hedge is perfect.

The Unified Model brings this real world problem to the forefront. It is not possible to hedge continuously. Traders decide when and how much to hedge based on available information. The job of mathematical finance practitioners is to help them figure out when (τ_j) and how (Γ_j) to adjust their hedge.

This model does not provide a solution, only a framework for a rigorous mathematical approach to understanding how to value, hedge, and manage the risk involved with trading market instruments under realistic conditions.