A SIMPLE PROOF OF THE FUNDAMENTAL THEOREM OF ASSET PRICING

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Abstract. A simple statement and accessible proof of a version of the Fundamental Theorem of Asset Pricing in discrete time is provided. Careful distinction is made between prices and cash flows in order to provide uniform treatment of all instruments. There is no need for a "real-world" measure in order to specify a model for derivative securities, one simply specifies an arbitrage free model, tunes it to market data, and gets down to the business of pricing, hedging, and managing the risk of derivative securities.

1. Introduction

It is difficult to write a paper about the Fundamental Theorem of Asset Pricing that is longer than the bibliography required to do justice to the excellent work that has been done elucidating the key insight Fischer Black, Myron Scholes, and Robert Merton had in the early '70's. At that time, the Capital Asset Pricing Model and equilibrium reasoning dominated the theory of security valuation so the notion that the relatively weak assumption of no arbitrage could have such detailed implications about possible prices resulted in well deserved Nobel prizes.

One aspect of the development of the FTAP has been the technical difficulties involved in providing rigorous proofs and the the increasingly convoluted statements of the theorem. The primary contribution of this paper is a statement of the fundamental theorem of asset pricing that is comprehensible to traders and risk managers and a proof that is accessible to students at graduate level courses in derivative securities. Emphasis is placed on distinguishing between prices and cash flows in order to give a unified treatment of all instruments. No artificial "real world" measures which are then changed to risk-neutral measures needed. (See also Biagini and Cont [4].) One simply finds appropriate price deflators.

Section 2 gives a brief review of the history of the FTAP with an eye to demonstrating the increasingly esoteric mathematics involved. Section 3 states and proves the one period version and introduces a definition of arbitrage more closely suited to what practitioners would recognize. Several examples are presented to illustrate the usefulness of the theorem. In section 4 the general result for discrete time models is presented together with more examples. The last section finishes with some general remarks and a summary of the methodology proposed in this paper. The appendix is an attempt to clairify attribution of early results.

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Peter Carr is entirely responsible for many enjoyable and instructive discussions on this topic. Andrew Kalotay provided background on Edward Thorpe and his contributions. Alex Mayus provided practitioner insights. Robert Merton graciously straightened me out on the early history. Walter Schachermeyer provided background on the technical aspects of the state of the art proofs. I am entirely responsible for any omissions and errors.

2. Review

From Merton's 1973 [25] paper, "The manifest characteristic of (21) is the number of variables that it does *not* depend on" where (21) refers to the Black-Scholes 1973 [5] option pricing formula for a call having strike E and expiration τ

$$f(S,\tau;E) = S\Phi(d_1) - Ee^{-rt}\Phi(d_1 - \sigma\sqrt{\tau}).$$

Here, Φ is the cumulative standard normal distribution, σ^2 is the instantaneous variance of the return on the stock and $d_1 = [\log(S/E) + (r + \frac{1}{2}\sigma^2)\tau]/\sigma\sqrt{\tau}$. In particular, the return on the stock does not make a showing, unlike in the Capital Asset Pricing Model where it shares center stage with covariance. This was the key insight in the connection between arbitrage-free models and martingales.

In the section immediately following Merton's claim he calls into question the rigor of Black and Scholes' proof and provides his own. His proof requires the bond process to have nonzero quadratic variation. Merton 1974 [26] provides what is now considered to be the standard derivation.

A special case of the valuation formula that European option prices are the discounted expected value of the option payoff under the risk neutral measure makes its first appearance in the Cox and Ross 1976 [7] paper. The first version of the FTAP in a form we would recognize today occurs in a Ross 1978 [31] where it is called the Basic Valuation Theorem. The use of the Hahn-Banach theorem in the proof also makes its first appearance here, although it is not clear precisely what topological vector space is under consideration. The statement of the result is also couched in terms of market equilibrium, but that is not used in the proof. Only the lack of arbitrage in the model is required.

Harrison and Kreps [13] provide the first rigorous proof of the one period FTAP (Theorem 1) in a Hilbert space setting. They are also the first to prove results for general diffusion processes with continuous, nonsingular coefficients and make the premonitory statement "Theorem 3 can easily be extended to this larger class of processes, but one then needs quite a lot of measure theoretic notation to make a rigorous statement of the result."

The 1981 paper of Harrison and Pliska [14] is primarily concerned with models in which markets are complete (Question 1.16), however they make the key observation, "Thus the parts of probability theory most relevant to the general question (1.16) are those results, usually abstract in appearance and French in origin, which are invariant under substitution of an equivalent measure." This observation applies equally to incomplete market models and seems to have its genesis in the much earlier work of Kemeny 1955 [19] and Shimony 1955 [33] as pointed out by W. Schachermeyer.

D. Kreps 1981 [21] was the first to replace the assumption of no arbitrage with that of no *free lunch*: "The financial market defined by (X,τ), M, and π admits a free lunch if there are nets $(m_\alpha)_{\alpha\in I} \in M_0$ and $(h_\alpha)_{\alpha\in I} \in X_+$ such that $\lim_{\alpha\in I}(m_\alpha - h_\alpha) = x$ for some $x \in X_+\setminus\{0\}$." It is safe to say the set of traders and risk managers that are able to comprehend this differs little from the empty set. It was a brilliant technical innovation in the theory but the problem with first assuming a measure for the paths instrument prices follow was that it made it difficult to apply the Hahn-Banach theorem. The dual of $L^\infty(\tau)$ under the norm topology is intractable. The dual of $L^\infty(\tau)$ under the weak-star topology is $L^1(\tau)$, which by the Radon-Nikodym theorem can be identified with the set of measures that are absolutely continuous with respect to τ. This is what one wants when hunting for equivalent martingale measures, however one obstruction to the proof is that the positive functions in $L^\infty(\tau)$ do not form a weak-star open set. Krep's highly technical free lunch definition allowed him to use the full plate of open sets available in the norm topology that is required for a rigorous application of the Hahn-Banach theorem.

The escalation of technical machinery continues in Dalang, Morton and Willinger 1990 [8]. This paper gives a rigorous proof of the FTAP in discrete time for an arbitrary probability space and is closest to this paper in subject matter. They correctly point out an integrability condition on the price process is not economically meaningful since it is not invariant under change of measure. They give a proof that does not assume such a condition by invoking a nontrivial measurable selection theorem. They also mention, "However, if in addition the process were assumed to be bounded, ..." and point out how this assumption could simplify their proof. The robust arbitrage definition and the assumption of bounded prices is also used the original paper, Long Jr. 1990 [17], on numeraire portfolios.

The pinnacle of abstraction comes in Delbaen and Schachermeyer 1994 [9] where they state and prove the FTAP in the continuous time case. Theorem 1.1 states an equivalent martingale measure exists if and only if there is *no free lunch with vanishing risk*: "There should be no sequence of final payoffs of admissible integrands, $f_n = (H^n \cdot S)_\infty$, such that the negative parts f_n^- tend to zero uniformly and such that f_n tends almost surely to a $[0,\infty]$-valued function f_0 satisfying $P[f_0 > 0] > 0$." The authors were completely correct when they claim "The proof of Theorem 1.1 is quite technical..."

The fixation on change of measure and market completeness resulted in increasingly technical definitions and proofs. This paper presents a new version of the Fundamental Theorem of Asset Pricing in discrete time. No artificial probability measures are introduced and no "change of measure" is involved. The model allows for negative prices and for cash flows (e.g., dividends, coupons, carry, etc.) to be associated with instruments. All instruments are treated on an equal basis and there is no need to assume the existence of a risk-free asset that can be used to fund trading strategies.

As is customary, perfect liquidity is assumed: every instrument can be instantaneously bought or sold in any quantity at the given price. What is not customary is that prices are bounded and there is no a priori measure on the space of possible outcomes. The algebras of sets that represent available information determine the price dynamics that are possible in an arbitrage-free model.

3. The One Period Model

The one period model is described by a vector, $x \in \mathbf{R}^m$, representing the prices of m instruments at the beginning of the period, a set Ω of all possible outcomes over the period, and a bounded function $X\colon \Omega \to \mathbf{R}^m$, representing the prices of the m instruments at the end of the period depending on the outcome, $\omega \in \Omega$.

Definition 3.1. *Arbitrage exists if there is a vector $\gamma \in \mathbf{R}^m$ such that $\gamma \cdot x < 0$ and $\gamma \cdot X(\omega) \geq 0$ for all $\omega \in \Omega$.*

The cost of setting up the position γ is $\gamma \cdot x = \gamma_1 x_1 + \cdots + \gamma_m x_m$. This being negative means money is made by putting on the position. When the position is liquidated at the end of the period, the proceeds are $\gamma \cdot X$. This being non-negative means no money is lost.

It is standard in the literature to introduce an arbitrary probability measure on Ω and use the conditions $\gamma \cdot x = 0$ and $\gamma \cdot X \geq 0$ with $E[\gamma \cdot X] > 0$ to define an arbitrage opportunity, e.g., Shiryaev, Kabanov, Kramkov and Melnikov [34] section 7.3, definition 1. Making nothing when setting up a position and having a nonzero probability of making a positive amount of money with no estimate of either the probability or amount of money to be made is not a realistic definition of an arbitrage opportunity. Traders want to know how much money they make up-front with no risk of loss after the trade is put on. This is what Garman [12] calls *strong arbitrage*.

Define the *realized return* for a position, γ, by $R_\gamma = \gamma \cdot X / \gamma \cdot x$, whenever $\gamma \cdot x \neq 0$. If there exists $\zeta \in \mathbf{R}^m$ with $\zeta \cdot X(\omega) = 1$ for $\omega \in \Omega$ (a zero coupon bond) then the price is $\zeta \cdot x = 1/R_\zeta$. Zero interest rates correspond to a realized return of 1.

Note that arbitrage is equivalent to the condition $R_\gamma < 0$ on Ω for some $\gamma \in \mathbf{R}^m$. In particular, negative interest rates do not necessarily imply arbitrage.

The set of all arbitrages form a cone since this set is closed under multiplication by a positive scalar and addition. The following version of the FTAP shows how to compute an arbitrage when it exists.

Theorem 3.1. *(One Period Fundamental Theorem of Asset Pricing) Arbitrage exists if and only if x does not belong to the smallest closed cone containing the range of X. If x^* is the nearest point in the cone to x, then $\gamma = x^* - x$ is an arbitrage.*

Proof. If x belongs to the cone, it is arbitrarily close to a finite sum $\sum_j X(\omega_j)\pi_j$, where $\omega_j \in \Omega$ and $\pi_j > 0$ for all j. If $\gamma \cdot X(\omega) \geq 0$ for all $\omega \in \Omega$ then $\gamma \cdot \sum X(\omega_j)\pi_j \geq 0$, hence $\gamma \cdot x$ cannot be negative. The other direction is a consequence of the following with $\mathcal{C}$ being the smallest closed cone containing $X(\Omega)$. □

Lemma 3.2. *If $\mathcal{C} \subset \mathbf{R}^m$ is a closed cone and $x \notin \mathcal{C}$, then there exists $\gamma \in \mathbf{R}^m$ such that $\gamma \cdot x < 0$ and $\gamma \cdot y \geq 0$ for all $y \in \mathcal{C}$.*

Proof. This result is well known, but here is an elementary self-contained proof. Since $\mathcal{C}$ is closed and convex, there exists $x^* \in \mathcal{C}$ such that $\|x^* - x\| \leq \|y - x\|$ for all $y \in \mathcal{C}$. We have $\|x^* - x\| \leq \|tx^* - x\|$ for $t \geq 0$, so $0 \leq (t^2 - 1)\|x^*\|^2 - 2(t-1)x^* \cdot x = f(t)$. Because $f(t)$ is quadratic in t and vanishes at $t = 1$, we have $0 = f'(1) = 2\|x^*\|^2 - 2x^* \cdot x$, hence $\gamma \cdot x^* = 0$. Now $0 < \|\gamma\|^2 = \gamma \cdot x^* - \gamma \cdot x$, so $\gamma \cdot x < 0$.

Since $\|x^* - x\| \leq \|ty + x^* - x\|$ for $t \geq 0$ and $y \in \mathcal{C}$, we have $0 \leq t^2\|y\|^2 + 2ty \cdot (x^* - x)$. Dividing by t and setting $t = 0$ shows $\gamma \cdot y \geq 0$. □

Let $B(\Omega)$ be the Banach algebra of bounded real-valued functions on Ω. Its dual, $B(\Omega)^* = ba(\Omega)$, is the space of finitely additive measures on Ω, e.g., Dunford and Schwartz [11]. If $\mathcal{P}$ is the set of non-negative measures in $ba(\Omega)$, then $\{\langle X, \Pi\rangle : \Pi \in \mathcal{P}\}$ is the smallest closed cone containing the range of X, where the angle brackets indicate the dual pairing. There is no arbitrage if and only if there exists a non-negative finitely additive measure, Π, on Ω such that $x = \langle X, \Pi\rangle$. We call such Π a *price deflator*.

If $V \in B(\Omega)$ is the payoff function of an instrument and $V = \gamma \cdot X$ for some $\gamma \in \mathbf{R}^m$, then the cost of replicating the payoff is $\gamma \cdot x = \langle \gamma \cdot X, \Pi\rangle = \langle V, \Pi\rangle$. Of course the dimension of such perfectly replicating payoff functions can be at most

m. The second fundamental theorem of asset pricing states that when there are complete markets, the price is unique. But that never happens in the real world.

If a zero coupon bond, $\zeta \in \mathbf{R}^m$, exists then the riskless realized return is $R = R_\zeta = 1/\Pi(\Omega)$. If we let $P = \Pi R$, then P is a probability measure and $x = \langle X/R, P\rangle = EX/R$. With V as in the previous paragraph, the cost of the replicating payoff is $v = EV/R$, the expected discounted payoff.

3.0.1. *Managing Risk.* The current theoretical foundations of Risk Mangagment are lacking [1]. The classical theory assumes complete markets and perfect hedging and fails to provide useful tools for quantitatively assessing how wishful this thinking is.

The main defect of most current risk measures is that they fail to take into account active hedging. E.g., VaR[18] assumes trades will be held to some time horizon and only considers a percentile loss. The only use to someone running a business that they might lose X in n days with probability p if they do nothing is to put a tick in a regulatory checkbox.

Multi-period models will be considered below, but a first step is to measure the least squared error in the one-period model. Given any measure Π and any payoff $V \in B(\Omega)$, we can minimize $\langle(\gamma\cdot X - V)^2, \Pi\rangle$. The solution is $\gamma = \langle XX^T, \Pi\rangle^{-1}\langle XV, \Pi\rangle$. The least squared error is

$$\min_\gamma \langle(\gamma\cdot X - V)^2, \Pi\rangle = \langle V^2, \Pi\rangle - \langle XV, \Pi\rangle^T \langle XX^T, \Pi\rangle^{-1}\langle XV, \Pi\rangle.$$

In the case of a two instrument market $X = (R, S)$ where R is the realized return on a zero coupon bond we get $\gamma = ((EV - nES)/R, n)$ where $n = \operatorname{Cov}(S, V)/\operatorname{Var} S$ and the expectation corresponds to the probability measure $P = \Pi R$. If we further assume $x = (1, s)$ we have $\gamma\cdot x = EV/R - n(ES/R - s)$ and the least squared error reduces to $\sin^2\theta \operatorname{Var}(V)/R$ where $\cos\theta$ is the correlation of S with V.

If Π is a price deflator we get the same answer for the price as in the one-period model without the need to involve the Hahn-Banach theorem.

3.1. **Examples.** This section illustrates consequences of the one period model. Standard results follow from rational application of mathematics instead of ad hoc arguments.

Example 1. *(Put-Call parity) Let* $\Omega = [0, \infty)$, $x = (1, s, c, p)$, *and* $X(\omega) = (R, \omega, (\omega - k)^+, (k - \omega)^+)$.

This models a bond with riskless realized return R, a stock that can take on any non-negative value, and a put and call with the same strike. Take $\gamma = (-k/R, 1, -1, 1)$. Since $\gamma\cdot X(\omega) = -k + \omega - (\omega - k)^+ + (k - \omega)^+ = 0$ it follows $0 = x\cdot\gamma = -k/R + s - c + p$ so $s - k/R = c - p$.

This is the first thing traders check with any European option model. Put-call parity does not hold in general for American options because the optimal exercise time for each option is not necessarily the same.

Example 2. *(Cost of Carry) Let* $\Omega = [0, \infty)$, $x = (1, s, 0)$, *and* $X(\omega) = (R, \omega, \omega - f)$.

[1] As empirically verified in September 2008

This models a bond with riskless realized return R, a stock, and a forward contract on the stock with forward f. The smallest cone containing the range of X is spanned by $X(0) = (R, 0, -f)$ and $\lim_{\omega\to\infty} X(\omega)/\omega = (0, 1, 1)$. Solving $(1, s) = a(R, 0) + b(0, 1)$ gives $a = 1/R$ and $b = s$. This implies $0 = -f/R + s$ so $f = Rs$.

Example 3. *(Standard Binomial Model) Let $\Omega = \{d, u\}$, $0 < d < u$, $x = (1, s, v)$ and $X(\omega) = (R, s\omega, V(s\omega))$, where V is any given function.*

This is the usual (MBA) parametrization for the one period binomial model with a risk-less bond having realized return R, and a stock having price s that can go to either sd or su. The smallest cone containing the range of X is spanned by $X(d)$ and $X(u)$. Solving $(1, s) = aX(d) + bX(u)$ for a and b yields $a = (u - R)/R(u - d)$ and $b = (R - d)/R(u - d)$. The condition that a and b are non-negative implies $d \le R \le u$. The no arbitrage condition on the third component implies

$$v = \frac{1}{R}\left(\frac{u - R}{u - d}V(sd) + \frac{R - d}{u - d}V(su)\right).$$

In a binomial model, the option is a linear combination of the bond and stock. This is obviously a serious defect in the model. Solving $V(sd) = mR + nsd$ and $V(su) = mR + nsu$ for n we see the number of shares of stock to purchase in order to replicate the option is $n = (V(su) - V(sd))/(su - sd)$. Note that if V is a call spread consisting of long one call with strike slightly greater than sd and short one call with strike slightly less than su, then $\partial v/\partial s = 0$ since $V'(sd) = 0 = V'(su)$.

Example 4. *(Binomial Model) Let $\Omega = \{S^+, S^-\}$, $x = (1, s, v)$, and $X(\omega) = (R, \omega, V(\omega))$, where V is any given function.*

As above we find

$$v = \frac{1}{R}\left(\frac{S^+ - Rs}{S^+ - S^-}V(S^-) + \frac{Rs - S^-}{S^+ - S^-}V(S^+)\right)$$

and the number of shares of stock required to replicate the option is $n = (V(S^+) - V(S^-)/(S^+ - S^-)$. Note $\partial v/\partial s = n$ indicates the number of stock shares to buy in order to replicate the option.

Example 5. *Let $\Omega = [90, 110]$, $x = (1, 100, 6)$, and $X(\omega) = (1, \omega, \max\{\omega - 100, 0\})$.*

This corresponds to zero interest rate, a stock having price 100 that will certainly end with a price in the range 90 to 110, and a call with strike 100. One might think the call could have any price between 0 are 10 without entailing arbitrage, but that is not the case.

This model is not arbitrage free. The smallest cone containing the range of X is spanned by $X(90)$, $X(100)$, and $X(110)$. It is easy to see that x does not belong to this cone since it lies above the plane determined by the origin, $X(90)$ and $X(110)$.

Using e_b, e_s, and e_c as unit vectors in the bond, stock, and call directions, $X(90) = e_b + 90e_s$ and $X(110) = e_b + 110e_s + 10e_c$. Grassmann algebra[28] yields $X(110) \wedge X(90) = 90e_b \wedge e_s + 110e_s \wedge e_b + 10e_c \wedge e_b + 900e_c \wedge e_s = -900e_s \wedge e_c + 10e_c \wedge e_b - 20e_b \wedge e_s$. The vector perpendicular to this is $-900e_b + 10e_s - 20e_c$.

After dividing by 10, we can read off an arbitrage from this: borrow 90 using the bond, buy one share of stock, and sell two calls. The amount made by putting on this position is $-\gamma \cdot x = 90 - 100 + 12 = 2$. At expiration the position will be liquidated to pays $\gamma \cdot X(\omega) = -90 + \omega - 2\max\{\omega - 100, 0\} = 10 - |100 - \omega| \geq 0$ for $90 \leq \omega \leq 110$.

Example 6. *Let* $\Omega = [90, 110]$, $x = (100, 9.1)$, *and* $X(\omega) = (\omega, \max\{\omega - 100, \})$.

Eliminating the bond does not imply the call can have any price between 0 and 10 without arbitrage. The position $\gamma = (1, -11)$ is an arbitrage.

Example 7. *(Normal Model) Let* $\Omega = (-\infty, \infty)$, $x = (1, s)$, $X = (R, S)$ *with* R *scalar, and* S *normally distributed.*

This model was developed by Louis Bachelier in his 1900 PhD Thesis[1] with an implicit dependence on R. Choose the parameterization $S = Rs(1 + \sigma Z)$ where where Z is standard normal and the price deflator is $\Pi = P/R$ where P is the probability measure underlying Z. This model is arbitrage free for any value of σ, however it does allow for negative stock values. As long as σ is much smaller than s the probability of negative prices is negligible. Every model has its limitations.

A useful formula is $\mathrm{Cov}(N, f(M)) = \mathrm{Cov}(N, M)Ef'(M)$ whenever M and N are jointly normal. This follows from $Ee^{\alpha N}f(M) = Ee^{\alpha N}Ef(M + \alpha\,\mathrm{Cov}(M, N))$, taking a derivative with respect to α, then setting $\alpha = 0$.

The price of a put option with strike k is

$$\begin{aligned}
p(k) &= E(k - S)^+/R \\
&= E(k - S)1(S \leq k)/R \\
&= (k/R)P(S \leq k) - (ES/R)1(S \leq k) \\
&= (k/R - s)P(S \leq k) + (\mathrm{Var}(S)/R)E\delta_k(S)
\end{aligned}$$

since $d1(k - s)^+/ds = -\delta_k(s)$, where δ_k is a delta function with unit mass at k.

Let $\phi(z) = e^{-z^2/2}/\sqrt{2\pi}$ be the standard normal density and $\Phi(z) = \int_{-\infty}^{z} \phi(z)\,dz$ be the cumulative standard normal distribution. We have $E\delta_k(S) = E\delta_k(Rs(1 + \sigma Z)) = \phi(z)/Rs\sigma$ where $z = (k/Rs - 1)/\sigma$ hence $p(k) = (k/R - s)\Phi(z) + s\sigma\phi(z)$. For an at-the-money option, $k = Rs$, this reduces to $p(k) = s\sigma/\sqrt{2\pi}$.

The hedge position in the underlying is $\partial p(k)/\partial s = -ER1(S \leq k)/R = -\Phi(z)$ so the at-the-money hedge is to short $1/2$ share of stock.

For a general European option with payoff p we have the delta hedge is $\mathrm{Cov}(S, f(S))/\mathrm{Var}(S) = Ep'(S)$. If p is linear then we can find a perfect hedge so let's estimate the least squared error for quadratic payoffs. Letting $\mu_k = E(S - f)^k$ be the k-th central moment, where $f = Rs = ES$, and using $EZ^2 = 1$ and $EZ^4 = 3$ we find

$$\begin{aligned}
\mathrm{Var}(p(S)) &= \mu_2 p'(f)^2 + (\mu_4 - \mu_2^2)p''(f)^2/4 \\
&= f^2\sigma^2 p'(f)^2 + f^4\sigma^4 p''(f)^2/2.
\end{aligned}$$

Since $\mathrm{Cov}(S, p(S)) = \mathrm{Var}(S)Ep'(S) = \mathrm{Var}(S)p'(f)$ we have

$$\begin{aligned}
\mathrm{corr}(S, p(S)) &= 1/\sqrt{1 + f^2\sigma^2 p''(f)^2/2p'(f)^2} \\
&\approx 1 - f^2\sigma^2 p''(f)^2/4p'(f)^2
\end{aligned}$$

if $p'(f) > 0$ so $\sin\theta \approx f\sigma p''(f)/2p'(f)$ for small σ. The least squared error is $\operatorname{Var}(p(S))\sin^2\theta/R \approx f^2\sigma^2 p''(f)^2/4R$ which is second order in σ and does not depend (strongly) on $p'(f)$.

If $p'(f) = 0$ then the correlation is zero and the the best hedge is a cash position equal to $Ep(S)$. If $p'(f) < 0$ a similar estimate holds for the correlation tending to -1.

A curious result is that the at-the-money correlation for a call is constant: $\operatorname{corr}(S, (S-f)^+) = 1/\sqrt{2 - 2/\pi} \approx 0.856$ independent of R, s, and σ. This follows from $\operatorname{Cov}(S, p(S)) = \operatorname{Var}(S)/2$ and $\operatorname{Var} p(S) = \operatorname{Var}(S)(1/2 - 1/2\pi)$ where $p(x) = (x-f)^+$.

One technique traders use to smooth out gamma for at-the-money options is to extend the option expiration by a day or two. This gives a quantitative estimate of how bad that hedge might be.

3.2. **An Alternate Proof.**

The preceding proof of the fundamental theorem of asset pricing does not generalized to multi-period models.

Define $A\colon \mathbf{R}^m \to \mathbf{R} \oplus B(\Omega)$ by $A\xi = -\gamma\cdot x \oplus \gamma\cdot X$. This linear operator represents the account statements that would result from putting on the position γ at the beginning of the period and taking it off at the end of the period. Define $\mathcal{P}$ to be the set of $\{p \oplus P\}$ where $p > 0$ is in $\mathbf{R}$ and $P \geq 0$ is in $B(\Omega)$. Arbitrage exists if and only if $\operatorname{ran} A = \{A\gamma : \gamma \in \mathbf{R}^m\}$ meets $\mathcal{P}$. If the intersection is empty, then by the Hahn-Banach theorem [2] there exists a hyperplane $\mathcal{H}$ containing $\operatorname{ran} A$ that does not intersect $\mathcal{P}$. Since we are working with the norm topology, clearly $1 \oplus 1$ is the center of an open ball contained in $\mathcal{P}$, so the theorem applies. The hyperplane consist of all $y \oplus Y \in \mathbf{R} \oplus B(\Omega)$ such that $0 = y\pi + \langle Y, \Pi\rangle$ for some $\pi \oplus \Pi \in \mathbf{R} \oplus ba(\Omega)$.

First note that $\langle \mathcal{P}, \pi \oplus \Pi\rangle$ cannot contain both positive and negative values. If it did, the convexity of $\mathcal{P}$ would imply there is a point at which the dual pairing is zero and thereby meets $\mathcal{H}$. We may assume that the dual pairing is always positive and that $\pi = 1$. Since $0 = \langle A\gamma, \pi \oplus \Pi\rangle = \langle -\gamma\cdot x, \pi\rangle + \langle \gamma \cdot X, \Pi\rangle$ for all $\gamma \in \mathbf{R}^m$ it follows $x = \langle X, \Pi\rangle$ for the non-negative measure Π. This completes the alternate proof.

This proof does not yield the arbitrage vector when it exists, however it can be modified to do so. Define $\mathcal{P}^+ = \{\pi \oplus \Pi : \langle p \oplus P, \pi \oplus \Pi\rangle > 0, p \oplus P \in \mathcal{P}\}$. The Hahn-Banach theorem implies $\operatorname{ran} A \cap \mathcal{P} \neq \emptyset$ if and only if $\ker A^* \cap \mathcal{P}^+ = \emptyset$, where A^* is the adjoint of A and $\ker A^* = \{\pi \oplus \Pi : A^*(\pi \oplus \Pi) = 0\}$. If the later holds we know $0 < \inf_{\Pi \geq 0} \|-x + \langle X, \Pi\rangle\|$ since $A^*(\pi \oplus \Pi) = -x\pi + \langle X, \Pi\rangle$. The same technique as in the first proof can now be applied.

4. Multi-period Model

The multi-period model is specified by an increasing sequence of times $(t_j)_{0\leq j\leq n}$ at which transactions can occur, a sequence of algebras $(\mathcal{A}_j)_{0\leq j\leq n}$ on the set of possible outcomes Ω where $\mathcal{A}_j$ represents the information available at time t_j, a sequence of bounded $\mathbf{R}^m$ valued functions $(X_j)_{0\leq j\leq n}$ with X_j being $\mathcal{A}_j$ measurable that represent the prices of m instruments, and a sequence of bounded $\mathbf{R}^m$ valued functions $(C_j)_{1\leq j\leq n}$ with C_j being $\mathcal{A}_j$ measurable that represent the cash flows associated with holding one share of each instrument over the preceding time period. We further assume the cardinality of $\mathcal{A}_0$ is finite, and the $\mathcal{A}_j$ are increasing.

A *trading strategy* is sequence of bounded $\mathbf{R}^m$ valued functions $(\Gamma_j)_{0\le j\le n}$ with Γ_j being $\mathcal{A}_j$ measurable that represent the amount in each security purchased at time t_j. Your *position* is $\Xi_j = \Gamma_0 + \cdots + \Gamma_j$, the accumulation of trades over time. A trading strategy is called *closed out* at time t_j if $\Xi_j = 0$. Note in the one period case closed out trading strategies have the form $\Gamma_0 = \gamma$, $\Gamma_1 = -\gamma$.

The amount your *account* makes at time t_j is $A_j = \Xi_{j-1}\cdot C_j - \Gamma_j \cdot X_j$, $0 \le j \le n$, where we use the convention $C_0 = 0$. The financial interpretation is that at time t_j you receive cash flows based on the position held from t_{j-1} to t_j and are charged for trading Γ_j shares at prices X_j.

Definition 4.1. Arbitrage *exists if there is trading strategy that makes a strictly positive amount on the initial trade and non-negative amounts until it is closed out.*

We now develop the mathematical machinery required to state and prove the Fundamental Theorem of Asset Pricing.

Let $B(\Omega, \mathcal{A}, \mathbf{R}^m)$ denote the Banach algebra of bounded $\mathcal{A}$ measurable functions on Ω taking values in $\mathbf{R}^m$. We write this as $B(\Omega, \mathcal{A})$ when $m = 1$.

Recall that if B is a Banach algebra we can define the product $yy^* \in B^*$ for $y \in B$ and $y^* \in B^*$ by $\langle x, yy^*\rangle = \langle xy, y^*\rangle$ for $x \in B$, a fact we will use below.

The standard statement of the FTAP uses conditional expectation. This version uses restriction of measures, a much simpler concept. The conditional expectation of a random variable is defined by $Y = E[X|\mathcal{A}]$ if and only Y is $\mathcal{A}$ measurable and $\int_A Y\,dP = \int_A X\,dP$ for all $A \in \mathcal{A}$. Using the dual pairing this says $\langle 1_A Y, P\rangle = \langle 1_A X, P\rangle$ for all $A \in \mathcal{A}$. Using the product just defined we can write this as $\langle 1_A, YP\rangle = \langle 1_A, XP\rangle$ so $YP(A) = XP(A)$ for all $A \in \mathcal{A}$. If P has domain $\mathcal{A}$ this says $YP = XP|_{\mathcal{A}}$.

We need a slight generalization. If Y is $\mathcal{A}$ measurable, P has domain $\mathcal{A}$, and $\langle 1_A Y, P\rangle = \langle 1_A X, Q\rangle$ for all $A \in \mathcal{A}$, then $YP = XQ|_{\mathcal{A}}$. There is no requirement that P and Q be probability measures.

Let $\mathcal{P} \subset \bigoplus_{j=0}^n B(\Omega, \mathcal{A}_j)$ be the cone of all $\oplus_j P_j$ such that $P_0 > 0$ and $P_j \ge 0$, $1 \le j \le n$. The dual cone, $\mathcal{P}^+$ is defined to be the set of all $\oplus_j \Pi_j$ in $\bigoplus_{j=0}^n ba(\Omega, \mathcal{A}_j)$ such that $\langle P, \Pi\rangle = \langle \oplus_j P_j, \oplus_j \Pi_j\rangle = \sum_j \langle P_j, \Pi_j\rangle > 0$.

Lemma 4.1. *The dual cone $\mathcal{P}^+$ consists of $\oplus_j \Pi_j$ such that $\Pi_0 > 0$, and $\Pi_j \ge 0$ for $1 \le j \le n$.*

Proof. Since $0 < \langle P_0, \Pi_0\rangle$ for $P_0 > 0$ we have $\Pi_0(A) > 0$ for every atom of $\mathcal{A}_0$ so $\Pi_0 > 0$. For every $\epsilon > 0$ and any $j > 0$ we have $0 < \epsilon\Pi_0(\Omega) + \langle P_j, \Pi_j\rangle$ for every $P_j \ge 0$. This implies $\Pi_j \ge 0$. □

Theorem 4.2. *(Multi-period Fundamental Theorem of Asset Pricing) There is no arbitrage if and only if there exists $\oplus_i \Pi_i \in \mathcal{P}^+$ such that*

$$X_i \Pi_i = (C_{i+1} + X_{i+1})\Pi_{i+1}|_{\mathcal{A}_i}, \quad 0 \le i < n.$$

Note each side of the equation is a vector-valued measure and recall $\Pi|_{\mathcal{A}}$ denotes the measure Π restricted to the algebra $\mathcal{A}$.

Proof. Define $A\colon \bigoplus_{i=0}^n B(\Omega, \mathcal{A}_i, \mathbf{R}^m) \to \bigoplus_{i=0}^n B(\Omega, \mathcal{A}_i)$ by $A = \bigoplus_{0\le i\le n} A_i$. Define $\mathcal{C}$ to be the subspace of strategies that are closed out by time t_n.

With $\mathcal{P}$ as above, no arbitrage is equivalent to $A\mathcal{C} \cap \mathcal{P} = \emptyset$. Again, the norm topology ensures that $\mathcal{P}$ has an interior point so the Hahn-Banach theorem implies there exists a hyperplane $\mathcal{H} = \{X \in \bigoplus_{i=0}^n B(\Omega, \mathcal{A}_i) : \langle X, \Pi\rangle = 0\}$ for some $\Pi = \oplus_0^n \Pi_i$ containing $A\mathcal{C}$ that does not meet $\mathcal{P}$. It is not possible that $\langle \mathcal{P}, \Pi\rangle$ takes on different signs. Otherwise the convexity of $\mathcal{P}$ would imply $0 = \langle P, \Pi\rangle$ for some $P \in \mathcal{P}$ so we may assume $\Pi \in \mathcal{P}^+$. Note $0 = \langle A(\oplus_i \Gamma_i), \oplus_i \Pi_i\rangle = \sum_{i=0}^n \langle \Xi_{i-1} \cdot C_i - \Gamma_i \cdot X_i, \Pi_i\rangle$ for all $\oplus_i \Gamma_i \in \mathcal{C}$. Taking closed out strategies of the form $\Gamma_i = \Gamma$, $\Gamma_{i+1} = -\Gamma$ having all other terms zero yields, where Γ is $\mathcal{A}_i$ measurable, gives $0 = \langle \Xi_{i-1} \cdot C_i - \Gamma_i \cdot X_i, \Pi_i\rangle + \langle \Xi_i \cdot C_{i+1} - \Gamma_{i+1} \cdot X_{i+1}, \Pi_{i+1}\rangle = \langle -\Gamma \cdot X_i, \Pi_i\rangle + \langle \Gamma \cdot C_{i+1} + \Gamma \cdot X_{i+1}, \Pi_{i+1}\rangle$, hence $\langle \Gamma, X_i \Pi_i\rangle = \langle \Gamma, (C_{i+1} + X_{i+1})\Pi_{i+1}\rangle$ for all $\mathcal{A}_i$ measurable Γ. Taking Γ to be a characteristic function proves $X_i \Pi_i = (C_{i+1} + X_{i+1})\Pi_{i+1}|_{\mathcal{A}_i}$ for $0 \le i < n$. □

A simple induction shows

Corollary 4.3. *With notation as above,*

$$X_j \Pi_j = \sum_{j<i<k} C_i \Pi_i|_{\mathcal{A}_j} + (C_k + X_k)\Pi_k|_{\mathcal{A}_j}, \quad j < k. \tag{1}$$

This corrects and generalizes formula (2) in chapter 2 of Duffie [10]. As we will see below, this corollary is the primary tool for constructing arbitrage free models. In the case of zero cash flows and increasing algebras, the no arbitrage condition is equivalent to $(X_j \Pi_j)_{j \ge 0}$ being a martingale, by a slight abuse of the word martingale.

A standard way to define models is to specify a measure P on Ω and price deflators of the form $\Pi_i = D_i P$ for some $D_i \in B(\Omega, \mathcal{A}_i)$. In this case we can write $X_i \Pi_i = (C_{i+1} + X_{i+1})\Pi_{i+1}|_{\mathcal{A}_i}$ as $X_i D_i = E[(C_{i+1} + X_{i+1})D_{i+1}|\mathcal{A}_i]$.

In the one period case there is no need to distinguish between price and cash flows. In the multi-period case one can account for the cash flows, as in Pliska [29], by stipulating the price decreases by the amount of the cash flow. Explicitly distinguishing between prices and cash flows provides a unified model that uniformly incorporates other cash flows such as bond coupons or foreign exchange carry.

We say a closed strategy, Γ, is *self-financing* if all but the first and last component of $A\Gamma$ are zero. The cost at t_0 of creating the cash flow $\Xi_{n-1} \cdot C_n - \Gamma_n \cdot X_n$ at t_n is clearly $\Gamma_0 \cdot X_0 \Pi_0$.

Lemma 4.4. *If* (Γ_j) *is a closed out self-financing strategy then*

$$\langle \Gamma_0 \cdot X_0, \Pi_0\rangle = \langle \Xi_{n-1} \cdot C_n - \Gamma_n \cdot X_n, \Pi_n\rangle$$

.

Proof. First we show that $\langle \Gamma_0 \cdot X_0, \Pi_0\rangle = \langle \Xi_j \cdot X_j, \Pi_j\rangle$ for $j < n$. The result holds for $j = 0$. Assume it holds for j, then using the FTAP and self-financing condition

$$\begin{aligned}
\langle \Xi_j \cdot X_j, \Pi_j\rangle &= \langle \Xi_j \cdot (C_{j+1} + X_{j+1}), \Pi_{j+1}\rangle \\
&= \langle \Gamma_{j+1} \cdot X_{j+1} + \Xi_j \cdot X_{j+1}, \Pi_{j+1}\rangle \\
&= \langle \Xi_{j+1} \cdot X_{j+1}, \Pi_{j+1}\rangle.
\end{aligned}$$

Finally, $\langle \Xi_{n-1} \cdot X_{n-1}, \Pi_{n-1}\rangle = \langle \Xi_{n-1} \cdot (C_n + X_n), \Pi_n\rangle = \langle \Xi_{n-1} \cdot C_n - \Gamma_n \cdot X_n, \Pi_n\rangle$ since $\Xi_{n-1} = -\Gamma_n$ for closed strategies. □

This lemma shows that if a European derivative has payoff $V : \Omega \to \mathbf{R}$ at t_n and we can find a closed self-financing portfolio $(\Gamma_j)_{0 \le j < n}$ such that $\Xi_{n-1} \cdot C_n - \Gamma_n \cdot X_n = V$, then the cost of a the replicating strategy is $\langle V, \Pi_n/\Pi_0\rangle$. Since

$\Gamma_0 \cdot X_0 = \langle V, \Pi_n/\Pi_0 \rangle$ we can compute the initial hedge by taking the derivative with respect to market values $\Gamma_0 = (d/dX_0)\langle V, \Pi_n/\Pi_0 \rangle$.

This formula is the foundation of delta hedging derivative securities. In general such a strategy does not exist, but we could use an optimization criteron, e.g., best least squares fit, and use the fitting error as a measure of hedging risk.

4.1. **Examples.**

Example 8. *(Short Rate Process) A* short rate (realized return) process $(R_j)_{j\geq 0}$ *is a scalar valued adapted process that defines instruments having price* $X_j = 1$ *and a single non zero cash flow* $C_{j+1} = R_j$ *at time* t_{j+1}.

No arbitrage implies $\Pi_j = R_j \Pi_{j+1}|_{\mathcal{A}_j}$ so $R_j = \Pi_j/\Pi_{j+1}|_{\mathcal{A}_j}$. If the price deflators are predictable, i.e., Π_{j+1} is $\mathcal{A}_j$ measurable, $j \geq 0$, then $R_j = \Pi_j/\Pi_{j+1}$. In this case the short rate process determines the price deflators $\Pi_j = \Pi_0/(R_0 \cdots R_{j-1})$, $j > 0$.

Assuming price deflators are predictable is a tame assumption. It means that at any given time one can borrow or lend at a known rate over the next period. Note these can be used to guarantee self-financing strategies always exist.

This result is the foundation of fixed income derivatives. The price of all other fixed income derivatives (with no default) are constrained by the short rate process.

Example 9. *(Zero Coupon Bonds) A zero coupon bond has a single cash flow* $C_k = 1$ *at maturity* t_k.

Since $X_j \Pi_j = \Pi_k|_{\mathcal{A}_j}$ for a bond maturing at t_k we have its price at time $t_j \leq t_k$ is $X_j \equiv D_j(k) = \Pi_k/\Pi_j|_{\mathcal{A}_j} = \Pi_k|_{\mathcal{A}_j}/\Pi_j$. The price at and after maturity is 0. Note $D_j(j+1) = 1/R_j$. The function $j \mapsto D_0(j)$ is called the *discount* or *zero* curve.

Example 10. *(Forward Rate Agreement) A forward rate agreement starting at* t_i *has price* $X_i = 0$ *and two non-zero cash flows,* $C_j = -1$ *at* t_j *and* $C_k = 1 + F_i(j,k)\delta(j,k)$ *at* t_k *where* $\delta(j,k)$ *is the* daycount fraction *for the interval* $[t_j, t_k]$.

The *day count basis* (Actual/360, 30/360, etc.) is a market convention that determines the day count fraction and is approximately equal to the time in years of the corresponding interval.

We have $0 = -1\Pi_j|_{\mathcal{A}_i} + (1 + F_i(j,k)\delta(j,k)\Pi_k|_{\mathcal{A}_i}$ so

$$F_i(j,k) = \frac{1}{\delta(j,k)}\left(\frac{\Pi_j}{\Pi_k} - 1\right)|_{\mathcal{A}_i} = \frac{1}{\delta(j,k)}\left(\frac{D_i(j)}{D_i(k)} - 1\right).$$

Forward rates are determined by zero coupon bond prices since they are a portfolio of such.

Note that if a zero coupon bond with maturity t_k is available at time t_j then $F_j(j,k) = (1/D_j(k) - 1)/\delta(j,k)$ is the *forward rate* over the interval.

Example 11. *(Bonds) A bond is specified by* calculation dates $t_0 < t_1 < \cdots < t_n$, *cash flows* $C_j = c\delta_j$, $0 < j < n$, *and* $C_n = 1 + c\delta_n$ *where* $\delta_j = \delta(j-1, j)$.

The price at time t_0 satisfies $X_0\Pi_0 = c\sum_{j=1}^n \delta_j \Pi_j|\mathcal{A}_0 + \Pi_n|\mathcal{A}_0$ so $X_0 = c\sum_j \delta_j D_0(j) + D_0(n)$. A bond is priced at par if $X_0 = 1$ in which case $c = (1 - D_0(n))/\sum_j \delta_j D_0(j)$ is the *par coupon.*

Example 12. *(Swaps) A swap is specified by calculation dates* $t_0 < t_1 < \cdots < t_n$ *and cash flows* $C_j = (c - F_{j-1}(j-1, j))\delta_j$, $0 < j \leq n$

There are many types of swaps. This one is more accurately described as paying fixed and receiveing float without exchange of principal. It is also common for the day count basis of the fixed and floating legs to be different.

A fundamental fact about the floating cash flow stream is

$$\begin{aligned}\sum_{j=1}^n F_{j-1}(j-1,j)\delta_j \Pi_j|_{\mathcal{A}_0} &= \sum_{j=1}^n (\Pi_{j-1}/\Pi_j - 1)|_{\mathcal{A}_j}\Pi_j|_{\mathcal{A}_0} \\ &= \sum_{j=1}^n (\Pi_{j-1} - \Pi_j)|_{\mathcal{A}_j}|_{\mathcal{A}_0} \\ &= \Pi_0 - \Pi_n|_{\mathcal{A}_0}.\end{aligned}$$

This shows the value of the floating leg is the same as receiving a cash flow of 1 at t_0 and paying a cash flow of 1 at t_n. The intuition is that the initial cash flow can be invested at the prevailing forward rate over each interval and rolled over while harvesting the floating payments until maturity.

Swaps are typically issued at t_0 with price $X_0 = 0$. Using the above fact shows the swap par coupon is determined by the same formula as for a bond. More generally, if $X_t = 0$ for $t \le t_0$ and $X_t = 0$ we write

$$F_t^\delta(t_0, \dots, t_n) = \frac{D_t(t_0) - D_t(t_n)}{\sum_{j=1}^n \delta(j-1,j) D_t(t_j)}$$

for the par coupon at time t corresponding to the underlying (forward starting) swap. Note we are using the actual times instead of the index as arguments. Also note that a one period swap is simply a forward rate agreement.

Example 13. *(Futures) The price of a futures is always zero. Given an underlying index S_k at expiration t_k, they are quoted as having 'price' Φ_j at t_j with the constraint $\Phi_k = S_k$ at t_k. Their cash flows are $C_j = \Phi_j - \Phi_{j-1}$, $j \le k$.*

No arbitrage implies $0 = (\Phi_{j+1} - \Phi_j)\Pi_{j+1}|_{\mathcal{A}_j}$. If the deflators are predictable then $\Phi_j = \Phi_{j+1}|_{\mathcal{A}_j} = S_k|_{\mathcal{A}_j}$. The standard way of making this statement is to say futures quotes are a martingale.

If we assume there is a probability measure P on Ω such that $\Pi_t = D_t P$ for some D_t , the *stochastic discount* to time t, that are bounded $\mathcal{A}_t$ measurable functions then we can write $\langle X, \Pi_t \rangle = EXD_t$.

If F is a forward and D is the stochastic discount to expiration we have $0 = E(F - f)D = EFED + \operatorname{Cov}(F,D) - fED$ so the *convexity* is $\phi - f = -\operatorname{Cov}(F,D)/ED$, where $\phi = EF$ is the futures rate. In general F and D have negative correlation so futures quotes are higher than forward rates.

In the equity world it is often assumed the price deflators are not stochastic and $\Pi_t = D(0,t) \equiv D(t)$ is given. The *(instantaneous) spot rate*, $r(t)$, is defined by $D(t) = e^{-tr(t)}$ and the *(instantaneous) forward rate*, $f(t)$, by $D(t) = e^{-\int_0^t f(s)\,ds}$. We also write $D_s(t) = D(t)/D(s)$ for the discount from time s to t. Stock volatilities swamp out any dainty assumptions of stochastic rates.

Example 14. *(Generalized Ho-Lee Model*[16]*) The short rate process is $R_t = \phi(t) + \sigma(t)B_t$.*

The original Ho-Lee model specifies a constant volatility. It allows the discount curve to be fitted to market data. As in the Bachelier model, it allows interest rates to be negative, but it has a simple closed form solution using the fact

$\exp(-\int_0^t \Theta(s)^2/2\,ds + \int_0^t \Theta(s)\,dB_s)$ is a martingale plus $d(\Sigma(t)B_t) = \Sigma(t)\,dB_t + \Sigma'(t)B_t$ so $\int_t^u \sigma(s)B_s\,ds = \Sigma(u)B_u - \Sigma(t)B_t - \int_t^u \sigma(s)\,dB_s$ where $\Sigma' = \sigma$

$$\begin{aligned} E_t e^{-\int_t^u \sigma(s)B_s\,ds} &= E_t e^{-(\Sigma(u)B_u - \Sigma(t)B_t) + \int_t^u \Sigma(s)\,dB_s} \\ &= e^{-(\Sigma(u)B_t - \Sigma(t)B_t)} E_t e^{-(\Sigma(u)B_u - \Sigma(u)B_t) + \int_t^u \Sigma(s)\,dB_s} \\ &= e^{-(\Sigma(u)B_t - \Sigma(t)B_t)} E_t e^{\int_t^u (\Sigma(s)-\Sigma(u))\,dB_s} \\ &= e^{-(\Sigma(u)B_t - \Sigma(t)B_t)} e^{\frac{1}{2}\int_t^u (\Sigma(s)-\Sigma(u))^2\,ds} \end{aligned}$$

and E_t denotes conditional expectation with respect to time t. The generalized Ho-Lee model has discount prices

$$D_t(u) = e^{-\int_t^u \phi(s) - \frac{1}{2}(\Sigma(s)-\Sigma(u))^2\,ds + (\Sigma(u)-\Sigma(t))B_t}$$

where we reparameterize by replacing $\sigma(t)$ with $-\sigma(t)$. In case of constant volatility we have

$$D_t(u) = e^{-\int_t^u \phi(s) - \frac{1}{2}\sigma^2(s-u)^2\,ds + \sigma(u-t)B_t}.$$

This shows the convexity in the Ho-Lee model is $\phi(t) - f(t) = \frac{1}{2}\sigma^2 t^2$ which is quadratic in t.

Example 15. *(Forwards) A forward is a contract issued at time s and maturing at time t having price $X_s = 0$ and one nonzero cash flow $C_t = S_t - F_s(t)$ at time t, where S_t is the price at t of the underlying and $F_s(t)$ is the forward rate that is specified at time s.*

Assuming no dividends $S_sD(s) = S_tD(t)|_{\mathcal{A}_s}$ so $0 = (X_sD(s) = (S_t - F_s(t))D(t)|_{\mathcal{A}_s} = S_sD(s) - F_s(t))D(t)$ and we have $F_s(t) = S_s/D_s(t)$. This is just the cost-of-carry formula. In the presence of dividends (d_j) at (t_j) this formula becomes $F_s(t) = \sum_{s<t_j\le t} d_j|_{\mathcal{A}_s}/D_s(t_j) + S_s/D_s(t)$. Note dividends may be random.

Binomial models are based on a *random walk*. Let $\Omega = \{\omega = (\omega_1, \ldots, \omega_n) : \omega_j \in \{0,1\}, 1 \le j \le n\}$ and let P be the measure on Ω with $P(\{\omega\}) = 1/2^n$ for $\omega \in \Omega$. The equivalence relation $[\omega]_j = [\omega']_j$ if and only if $\omega_i = \omega'_i$ for $i \le j$ gives a partition that determines the atoms of the algebra $\mathcal{A}_j$.

Random walk is the discrete time stochastic process $W_j(\omega) = \omega_1 + \cdots + \omega_j$, $1 \le j \le n$. Note $P(W_j = k) = \binom{n}{k}/2^n$, $EW_j = j/2$, and $\mathrm{Var}(W_j) = j/2 - j^2/2$. If we let $Z_j = 2W_j - j$ then $EZ_j = 0$ and $\mathrm{Var}(Z_j) = j$.

Define $[\omega]_{j0} = [\omega]_{j+1} \cap \{\omega_{j+1} = 0\}$ and similarly for $[\omega]_{j1}$ so $[\omega]_j$ is the disjoint union of $[\omega]_{j0}$ and $[\omega]_{j1}$. It is easy to see $Z_{j+1}P|_{\mathcal{A}_j} = Z_jP$. More generally

$$\begin{aligned} f(Z_{j+1}(\omega))P([\omega]_j) &= f(Z_{j+1}(\omega))P([\omega]_{j0}) + f(Z_{j+1}(\omega))P([\omega]_{j1}) \\ &= f(Z_j(\omega) - 1)P([\omega]_j)/2 + f(Z_j(\omega)+1)P([\omega]_j)/2 \end{aligned}$$

so $f(Z_{j+1})P|_{\mathcal{A}_j} = \frac{1}{2}(f(Z_j - 1) + f(Z_j + 1))P$.

Example 16. *(Multi-period Binomial Model) Fix the annualized realized return $R > 0$, the initial stock price s, the drift μ, and the volatility σ. Define $X_j = (R_j, S_j) = (R^j, se^{\mu j + \sigma Z_j})$.*

Many price deflators exist but we will look for one having the form $\Pi_j = R^{-j}P$. Clearly $R^{j+1}\Pi_{j+1}|_{\mathcal{A}_j} = R^j\Pi_j$. Since $S_{j+1}\Pi_{j+1}|_{\mathcal{A}_j} = (e^\mu/R)\frac{1}{2}(e^{-\sigma}+e^\sigma)S_j\Pi_j$, the model is arbitrage free if $e^\mu = R/\cosh\sigma$.

Example 17. *(Geometric Brownian Motion) Fix the spot rate r, the initial stock price s the drift μ, and the volatility σ. Let B_t be standard Brownian motion and define $X_t = (e^{rt}, se^{\mu t+\sigma B_t})$.*

Let P be Brownian measure and recall $M_t^\lambda = e^{-\lambda^2 t/2+\lambda B_t}$ is a martingale. Looking for deflators of the form $e^{-rt}P$ ensures $e^{-rt}\Pi_t|_{\mathcal{A}_s} = e^{-rs}\Pi_s$. Since $S_t\Pi_t = se^{(\mu-r)t+\sigma B_t}$, the model is arbitrage free if $\mu = r-\sigma^2/2$.

The forward value of a put option paying $\max\{k-S,0\}$ at the end of the period is $E\max\{k-S,0\} = kP(S\le k) - ES1(S\le k) = kP(S\le k) - ESP(Se^{\sigma^2 t}\le k)$ where we use $Ee^N f(N) = Ee^N Ef(N+\operatorname{Var}(N))$. (More generally, $Ee^N f(N_1,\ldots) = Ee^N Ef(N+\operatorname{Cov}(N,N_1),\ldots)$ if N,N_1, ... are jointly normal.) This can be written $E\max\{k-S,0\} = kP(Z\le z) - fP(Z\le z-\sigma t)$ where $z = \sigma t/2 + (1/\sigma)\log k/f$, and $f = Rs$ is the forward price of the stock.

For a European option with payoff p at time t, the value of the option is $v = e^{-rt}Ep(S_t)$. The delta is

$$
\begin{aligned}
\partial v/\partial s &= e^{-rt}Ep'(S_t)e^{(r-\sigma^2/2)t+\sigma B_t}\\
&= Ep'(e^{\sigma^2 t}S_t)
\end{aligned}
$$

and the gamma is

$$
\begin{aligned}
\partial^2 v/\partial s^2 &= Ep''(e^{\sigma^2 t}S_t)e^{\sigma^2 t}e^{(r-\sigma^2/2)t+\sigma B_t}\\
&= e^{(r+\sigma^2)t}Ep''(e^{2\sigma^2 t}S_t).
\end{aligned}
$$

4.2. **Infinitely Divisible Distrbutions.** Brownian motion is characterized as a stochastic process having increments that are independent, stationary, and normally distributed. Dropping the last requirement characterizes Lévy processes[3]. Knowing the distribution at time 1 determines the distribution at all times and the distribution at any time is infinitely divisible.

Prior to Lévy and Khintchine, Kolmogorov [20] derived a parameterization for the characteristic function of infinitely divisible distributions having finite variance. There exists a number γ and a non-decreasing function $G(x)$ such that

$$
\log Ee^{iuX} = i\gamma u + \int_{-\infty}^{\infty} K_u(x)\,dG(x),
$$

where $K_u(x) = (e^{iux}-1-iux)/x^2$. Note $\phi'(u) = i\gamma + i\int_{-\infty}^{\infty}(e^{iux}-1)/x\,dG(x)$ and $\phi''(u) = -\int_{-\infty}^{\infty}e^{iux}\,dG(x)$ so $EX = -i\phi'(0) = \gamma$ and $\operatorname{Var}X = -\phi''(0) = \int_{-\infty}^{\infty}dG(x) = G(\infty)-G(-\infty)$.

Lemma 4.5. *If X is infinitely divisible with Kolmogorov parameters γ and G, then $Ee^{isX}e^{iuX} = Ee^{isX}Ee^{iuX^*}$ where X^* has Kolmogorov parameters $\gamma^* = \gamma + \int_{-\infty}^{\infty}(e^{isx}-1)/x\,dG(x) = -i\phi'(s)$ and $dG^*(x) = e^{isx}dG(x)$.*

Proof. We have

$$\begin{aligned} Ee^{isX}e^{iuX} &= Ee^{i\gamma(s+u)+\int_{-\infty}^{\infty} K_{s+u}(x)\,dG(x)} \\ &= Ee^{isX}e^{i\gamma u+\int_{-\infty}^{\infty}(K_{s+u}(x)-K_s(x))\,dG(x)} \end{aligned}$$

A simple calculation shows $K_{s+u}(x) - K_s(x) = iu(e^{isx} - 1)/x + e^{isx}K_u(x)$ so $Ee^{isX}e^{iuX} = Ee^{isX}Ee^{iuX^*}$ where X^* is infinitely divisible with Kolmogorov parameters $\gamma^* = -i\phi'(s)$ and $dG^*(x) = e^{isx}\,dG(x)$. □

We call X^* the K-transform of X.

If X is standard normal, then $\gamma = 0$, $G = 1_{[0,\infty)}$ and $\phi(u) = -u^2/2$ so $\gamma^* = is$ and $dG^* = dG$. We have $e^{-s^2/2}Ee^{iu(is+X)} = e^{-s^2/2}e^{-su-u^2/2} = e^{-(s+u)^2/2} = Ee^{isX}e^{iuX}$.

Corollary 4.6. *If f and its Fourier transform are integrable, then $Ee^{isX}f(X) = Ee^{isX}Ef(X^*)$ where X^* is the K-transform of X.*

Proof. If f and its Fourier transform are integrable, then $f(x) = \int_{-\infty}^{\infty} e^{iux}\hat{f}(u)\,du/2\pi$, where $\hat{f}(u) = \int_{-\infty}^{\infty} e^{-iux}f(x)\,dx$ is the Fourier transform of f.

$$\begin{aligned} Ee^{isX}f(X) &= \int_{-\infty}^{\infty} Ee^{iuX}e^{iuX}\hat{f}(u)\,du/2\pi \\ &= Ee^{isX}\int_{-\infty}^{\infty} Ee^{iuX^*}\hat{f}(u)\,du/2\pi \\ &= Ee^{isX}Ef(X^*) \end{aligned}$$

□

Example 18. *(Lévy Processes) Fix the spot rate r, the initial stock price s, the drift μ, and the volatility σ. Let L_t be a Lévy process and define $X_t = (e^{rt}, se^{\mu t+\sigma L_t})$.*

Again we look for deflators of the form $e^{-rt}P$. If we define the *cumulant* $\kappa_t(s) = \log Ee^{sL_t}$ then $\kappa_t(s) = t\kappa_1(s)$ and $e^{-t\kappa_1(\sigma)+\sigma L_t}$ is a martingale. Since $S_t\Pi_t = se^{(\mu-r)t+\sigma L_t}$, the model is arbitrage free if $\mu = r - \kappa_1(\sigma)$.

The formula for the forward value of put is $E(k - S_t)^+ = E(k - S_t)1(S_t \le k) = kP(S_t \le k) - se^{rt}P(S_t^* \le k)$ where $S_t^* = se^{(r-\kappa_1(\sigma))t+\sigma L_t^*}$ and L_t^* is the K-transform with $is = \sigma$.

5. Remarks

- Not only do traders want to know exactly how much they make upfront based on the size of the position they put on, they and their risk managers also want to hedge the subsequent gains they might make under favorable market conditions.
- Different counterparties have different short rate processes. A large financial institution can fund trading strategies at a more favorable rate than a day trader using a credit card.
- As previously noted, $\partial v/\partial s \ne n$ in Example 2, however $\partial(Rv)/\partial R = ns$ for both Example 2 and 3. In words, the derivative of the future value of the option with respect to realized return is the dollar delta.

- It is not necessay to assume algebras for the prices and cash flows are are increasing. If they are adapted to the algebras $(\mathcal{B}_j)$ and $\mathcal{B}_j \subseteq \mathcal{A}_j$ for all j then $X_j\Pi_j$ will be well defined. This is useful in order to model a recombining tree. In the standard binomial model the atoms of $\mathcal{B}_j$ are $\{W_j = j - 2i\}$, $0 \le i \le j$. This can be used to give a rigorous foundation to path bundling algorithms, e.g., Tilley [35].
- This theory only allows bounded functions as models of prices and positions. This corresponds to reality, but not to the classical Black-Scholes/Merton theory. The fact that prices are bounded has no material consequences when it comes to model implementation. An unbounded price process can be replaced by one stopped at an arbitrarily large value. Since we can make the probability of stopping vanishingly small, calculation of option prices can be made arbitrarily close to those computed using the unbounded model. Every model I have implemented had prices bounded by 1.8×10^{308}.
- Likewise, discrete time is not material problem since one could model yocto second time steps. In fact, continuous time introduces serious technical problems such as doubling strategies[13]. Zeno wasn't the only one to distract people's attention with this sort of casuistry.
- Measures being finitely additive is also not an issue. Countably additive measures are also finitely additive and so all such models fit into this framework. Interchanging limits and the Radon-Nikodym theorem for finitely additive measures are more complicated than for countably additive measures, but these are not needed here.
- The examples show this theory has the same expressive power as the standard theory and illustrates the usefulness of distinguishing prices from cash flows to uniformly handle all types of instruments. There is no need to cook up a "real world" measure. Not only does it ultimately get replaced, it adds technical complications to the theory.

6. Appendix: Origins

While preparing this paper I had difficulty understanding who figured out what when in the early theory. Cutting edge research is always messy. This appendix is my attempt to clear that up and point out the repercussions. Priority is the currency of academics, legacy is the other side of that coin.

Currency is both sides of the coin for practitioners and I make my living trying to provide them with tools they find useful. They usually don't understand the subtleties of mathematical models but they know if the software implementation provides numbers that make sense.

As George Box said "all models are wrong, but some are useful." Mathematical Finance is still in its infancy, but it has notched up some significant victories. Dollar denominated fixed income derivatives having maturity less than 4 years trade at basis point spreads. Every bank has a different implementation, but they all get the same answer. "Practitioners" in that market can no longer rely on cunning and makeshift.

As Haug and Taleb[15] carefully delineate, the Black-Scholes and even more sophisticated formulas were used well before Black, Scholes, and Merton showed up on the scene. They underscore the importance of the no arbitrage condition and are entirely correct that traders still use ad hoc devices to produce numbers they find useful. Options are used to determine model parameters and now play the role of primary securities in hedging more complex derivatives. Such is financial market progress.

However, they don't seem to appreciate the power of the mathematical underpinnings. Ed Thorpe came up with a formula for calls and puts, but didn't know how to extend that to price bonds with embedded options[2]. Academics have time to reflect on the paths blazed by practicioners. Exotic option pricing formulas require nontrivial mathematics unobtainable through seat-of-the-pants methods.

It is beyond the scope of this appendix to review the tenor of the time laid down by Markowitz[23], Tobin[36], Sharpe[32], Lintner[22] and other pioneers in the field of quantitative finance, but they developed an economic theory to quantify how diversification reduced risk. The Capital Asset Pricing Model showed how to create portfolios that could minimize systemic market risk.

Many of the fundamental results in the FTAP can be traced back to Merton's unpublished, but widely circulated, technical report[24] that ultimately became chapter 11 in his book on continuous time finance[27]. It uses a general equilibrium pricing model (intertemporal CAPM) to derive the Black-Scholes option model. His proof did not require normally distributed returns or a quadratic utility function, as CAPM did, foreshadowing Ross's Arbitrage Pricing Theory[30].

Merton also derived what is now called the Black-Scholes partial differential equation and showed how individual sample paths could be used to model prices directly instead of only considering expected values. Black and Scholes introduced the idea of dynamic trading when people were thinking in terms of portfolio selection. They showed continuous time trading with prices modeled by an Itō diffusion allows perfect replication and that the problem of estimating mean stock returns was irrelevant to pricing options.

This had some deleterious knock on effects in the theory of mathematical finance. Merton was so far ahead of his time with the mathematical tools he introduced that generations of people in his field overestimated the power of mathematics when it came to modeling the complicated world we live in. People that did not have his ability to understand the math latched on to binomial models. Brownian motion is a binomial model in wolves clothing.

Haug and Taleb are on the right track when it comes to pointing out the consequences of a theory that no practitioner would find plausible. I embarrassed myself in my early career when a trader asked me how to price a barrier option that was triggered on the second touch. For some reason he didn't buy my explanation about the infinite oscillatory behavior of Brownian motion and that even using the 100th time it touched the barrier would have the same theoretical price.

The work of Boyce and Kalotay [6] was far ahead of its time. They took a practical Operations Research approach to modeling what happens at the cash flow level, including counterparty credit and tax considerations. Something clumsily being rediscovered in our post September 2008 world.

The origin of the modern theory of derivative securities is based on Stephen Ross's 1977 paper "A Simple Approach to the Valuation of Risky Streams." He was the first to realize that the assumption of no arbitrage and the Hahn-Banach theorem placed a constraint on the dynamics of sample paths. It is a purely geometric result. The price deflator is simply a positive measure used to find a point in

[2]Andrew Kalotay, personal communication

a cone. Normalizing that to a probability measure does not tell you the probability of anything, although the normalizing factor does tell you the price of a zero coupon bond if your model has one.

Ross's approach was not as rigorous as Merton's and the attempts to place his results on sound mathematical footing led to the the escalation of increasingly abstract mathematical machinery outlined in the Review section. This paper is an endeavor to provide a statement of the fundamental theorem of asset pricing that practicioners can understand and a mathematically rigorous proof that is accessible to masters level students.

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